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Two-particle systems in relativistic Schrödinger theory

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Abstract. The relativistic Schrödinger theory (RST) is applied to a general 2-particle system with electromagnetic interactions. The emphasis of the investigation lies on the exchange and overlap effects in connection with the development of adequate single-particle and 2-particle concepts. It is possible to eliminate the 2-particle quantities and to formulate the 2-particle theory exclusively in terms of single-particle quantities but then there arises a new unobservable field degree of freedom: '*renormalization*'.

1. Introduction

The most intriguing features of the quantum world become evident when considering many-particle systems [1]. As is well known, these systems obey a rather different logic than is encountered in classical physics, but this point still remains the subject of violent controversies. (For a more philosophical survey of the relationship between the quantum and classical approaches see [2, 3].) Naturally, the treatment of many-particle systems requires the development of a framework embracing both single-particle and many-particle concepts. The many-particle effects occurring in a physical system are thought to be most effectively accounted for by the conventional quantum field theory (e.g. [4]). However, the latter theory is of essentially *statistical* character and therefore must live over some configuration space characterizing the kinematical properties of the system. But on the other hand, one could also think of a true space-time description of the quantum phenomena which would then necessarily be of a certain *fluid-dynamic* character. It seems that this latter line of thinking has not received much attention in comparison with the statistical approach, because the configuration-space description appears to be ideally adapted to those typically quantum effects as, e.g., the exchange phenomena which are kinematically based upon the symmetry under permutation of the particles.

However, it has recently been shown in some papers about the relativistic Schrödinger theory (RST) [5,6] that the notorious exchange effects of many-particle systems can, generally, also be treated in a quantum theory of fluid-dynamic character. Here, the discreteness of the particle number N_{p} is no problem because it agrees (roughly) with the number of dimensions of the typical fibre of the vector bundle to be applied. Thus one obtains a fluid-dynamic theory over the real space-time with an arbitrary discrete particle number N_{p}. Clearly, such a newcomer among the possible quantum theories cannot cope from the very beginning with the longstanding and well-established forms of quantum theory (i.e. the conventional quantum field theory) which have been developed by generations of physicists during the past decades. But what one must expect from a new theory is that, in its initial stage, it stands up to a certain

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number of elementary tests, which principally aim at its mathematical consistency and logical coherence. The present RST seems to have passed some of those tests with sufficient success: integrability conditions for the existence of wavefunctions [5], identities and conservation laws [7], topological currents [8], mixtures and pure states [9, 10] and some cosmological applications [11,12]. Besides these more mathematical aspects, the RST also seems to provide a new viewpoint on the 'non-local' effects in quantum theory (particle–wave duality [13,14]).

Therefore, it may be time now to take a closer look at the *many-particle* systems from the viewpoint of RST. This means that we have to study the overlap and exchange effects of RST which also occur in a similar way in standard theory whenever the wavefunctions of several particles are overlapping. Surprisingly enough, this field of problems then turns out to be intimately connected with another phenomenon whose analogue in conventional quantum theory is called *'renormalization'*. This means that in RST there also emerge a number of unobservable fields (*'amplitude fields'*) which acquire their physical meaning only by multiplying them with a *'renormalization'* factor. Through this construction, the RST is revealed as a competitor of the conventional form of many-particle systems. Consequently, it is of interest to see what new aspects are brought into this field by the new approach.

1.1. Probabilistic approach

Here, let us first reiterate the general belief that the ultimate theory must be the quantum field theory, e.g. QED and QCD; and that relativistic quantum mechanics (as a coupled system of Klein–Gordon and/or Dirac equations) can, at best, be considered as an approximation: see, e.g., [15–17]. As a natural consequence of this belief, one can proceed along two different routes in order to build up relativistic quantum mechanics for many-particle systems: (i) either one achieves this by applying certain approximations to quantum field theory [15, 18, 19], or (ii) one intuitively tries to generalize the well known 1-particle case of relativistic quantum mechanics to the many-particle situation [20–22]. Clearly, a nice consistency check for both routes would consist in the demonstration that they actually meet together at the same relativistic quantum mechanics; and indeed this has been achieved within the framework of the so-called *'manifestly covariant constraint formalism'* for Hamiltonian relativistic mechanics [23]. Nevertheless, there remain some open questions despite this relative success. For instance, the many-particle interaction potential occurring in the coupled set of relativistic wave equations is obtained by perturbation expansion of the Bethe–Salpeter kernel and thus always leads to instantaneous (albeit non-local) interactions. Despite this dubious instantaneity, the potentials are used to compute the mass spectra of mesons as quark–antiquark bound systems: see, e.g., [24]. But what would a non-instantaneous many-particle interaction potential look like? Observe here that a consistent decomposition of such an N-body potential into 2-particle contributions does not seem to be possible [16].

Moreover, there also remains a more conceptual difficulty which is due to the fact that both routes (i) and (ii) to relativistic quantum mechanics rely upon its probabilistic interpretation. Such an approach must necessarily interpret the physical state as a probability amplitude over the system's configuration space (i.e. space-time). As a consequence, the physical state for an N-particle system depends upon the $4N$ space-time coordinates of the particles $\psi = \psi(x_{1\mu} \dots \psi_{N\mu})$ and therefore depends upon the N time coordinates $t_1 \dots t_N$. However, the physical meaning of these *'observation times'* [23] t_a $(a = 1 \dots N)$ is obscure and is generally considered as a difficulty (if not deficiency) of the theory [16].

1.2. Fluid-dynamic approach

In this situation, the RST provides us with a completely alternative view upon the relativistic systems of fixed particle number N. The point of departure here is the *fluid-dynamic* interpretation of the 1-particle wave equations (as they are applied for the description of the Bose–Einstein condensates). As a consequence, the many-particle systems are not described by means of the tensor products of 1-particle states but rather by means of the Whitney sum of the single-particle bundles. Correspondingly the observable quantities of the theory are the *'physical densities'* over the real space-time and the difficulty with the probabilistic 'observation times' is avoided. Thus, in contrast to the probabilistic approach, the fluid-dynamic approach is truly relativistic (see the 1-particle mass eigenvalue problem in section 9) and therefore can be generalized to also include spin and gravitation within the framework of general relativity [10–12]. Furthermore, the investigation of electromagnetic many-particle interactions [6, 25] demonstrates their non-instantaneous nature. This implies that there can occur at most 3-body forces, namely when the *'overlap current'* due to any two particles generates a gauge potential A_μ which itself then acts upon the wavefunction of a third particle via minimal coupling. In this way the ambiguities of the N-body forces of the conventional quantum theory mentioned above are avoided. This is due to the fact that the Whitney *sum* construction is used for the fluid-dynamic description of many-particle systems (in place of using the tensor *product* of Hilbert spaces in the conventional probabilistic approach).

In this paper we restrict ourselves to the study of the *general* properties of 2-particle systems and the results are elaborated in the following arrangement.

First we present the general N-particle case in RST in order to introduce the basic concepts (section 2). Next, the observables for the 2-particle system are introduced (section 3). Then we recognize the emergence of an $SO(2)$ subbundle of the proper gauge group $U(1)\times U(1)$ for two particles. This construction makes sense because the typical frequency in the $SO(2)$ bundle is the difference of the two frequencies of the $U(1)$ bundles and thus is related to the transition effects (section 4). Afterwards the Hamiltonian dynamics is discussed which gives us the opportunity to introduce the exchange fields as the off-diagonal elements of the Hamiltonian $\mathcal{H}_\mu$. On closer inspection of these 2-particle fields, they are revealed to be the product of certain 1-particle fields referring to *different* particles. During this procedure there emerge the *'amplitude fields'* which are considered to be unphysical because they are fixed only up to some arbitrary scaling factor (section 5). Since this scaling factor can be changed arbitrarily, the theory acquires an additional unobservable field degree of freedom (*'renormalization'*).

However, the amplitude fields are 'almost' of physical character in the sense that when multiplied by some further unobservable fields (the *'renormalization factors'*) they actually coincide with the physical densities which are the true observables of theory. Thus, in place of dealing with the physical densities, one can also deal with the unobservable amplitude fields and renormalization factors whose dynamical equations directly follow from those of the physical densities (sections 6 and 7).

The exchange and overlap effects of the theory emerge by considering the conserved currents which consist of a single-particle contribution and an overlap part (section 7). By a careful examination of the density dynamics, one recognizes that the exchange fields always occur in certain combinations of their longitudinal and transverse parts. These combinations give rise to the introduction of two new 2-particle fields: the *exchange field strength* and the *anholonomy tensor field* which both are invariant with respect to renormalization and therefore must be considered as physical objects. However, in section 7 we show that these latter 2-particle fields can also be factorized into products of different 1-particle fields so that the whole theory can actually be reformulated by means of 1-particle objects which,

however, are dynamically coupled. The corresponding equations of motion exhibit the correct covariant transformation behaviour with respect to renormalization (section 8). Thus, it becomes evident that the price for dealing only with regular 1-particle objects is the emergence of renormalization. The latter is measured by the anholonomy tensor field so that for the holonomic case the renormalization degree of freedom must disappear.

In section 9, the energy (i.e. mass) eigenvalue problem for a single particle under the action of the Coulomb force is discussed in detail. This demonstrates the specific way in which the *non-relativistic energy* problem is generalized by RST to the *relativistic mass* problem.

2. General N-particle theory

In order to make it clear that RST actually is a rather general framework applying to any particle number N_p, we first want to present a brief sketch of this general structure. After this has been done, one can easily specialize the general formulae to some prescribed particle number, especially to the 2-particle systems ($N_\mathrm{p} = 2$) considered in this paper.

As in most of the modern theories for elementary particles (see e.g. [26]), the RST also relies upon the usual dualistic picture of the elementary physical process: matter and interaction. Consequently, the whole system under consideration will consist of two dynamically coupled subsystems, namely the matter subsystem and the subsystem of the interactions. Clearly, any one of the two subsystems must be described in terms of its own variables, i.e. *wavefunction* ψ (or *intensity matrix* $\mathcal{I}$) for the material subsystem and *gauge potential* $\mathcal{A}_\mu$ as carrier of the interactions. Let us first consider the material subsystem.

The wavefunction $\psi = \{\psi^A, A = 1 \dots N_\mathrm{f}\}$ is understood here to be an N_f-component section of some complex vector bundle over pseudo-Riemannian space-time as the base manifold. According to the type of interaction (i.e. choice of the gauge group), the N_f components of the section $\psi(x)$ group into certain subsets ('gauge multiplets') which are acted upon by the various irreducible representations of the gauge algebra. The number N_p of multiplets, or equivalently 'modes' of the matter field, is the *'particle number'* N_p. Especially, for the electromagnetic interactions the gauge group is the N_f-fold product $U(1) \times U(1) \times \cdots \times U(1)$ of the one-dimensional Abelian group $U(1)$ and therefore the number N_p of particles (i.e. matter modes) coincides here with the fibre dimension N_f.

Since RST is a fluid-dynamic approach to the quantum phenomena, the proper *observable* quantities of the theory are the *physical densities* $\Delta_i(x)$ ($i = 1 \dots N_\mathrm{f}^2$). They are generated by the wavefunction ψ (or intensity matrix $\mathcal{I}$, resp.) together with some Hermitian operator (δ_i, say) acting over the typical fibre of the vector bundle:

$$\Delta_i(x) = \bar{\psi}(x) \cdot \delta_i \cdot \psi(x) \tag{2.1a}$$

$$\Delta_i(x) = \mathrm{tr}(\mathcal{I}(x) \cdot \delta_i). \tag{2.1b}$$

Recall that for a pure state (i.e. $\mathcal{I} = \psi \otimes \bar{\psi}$), expressions (2.1*a*) and (2.1*b*) coincide, but otherwise (2.1*b*) is the more general one. The set of 'observables' $\{\delta_i\}$ may be grouped into representation spaces of the gauge group $\{\mathcal{S}\}$ so that the corresponding transformation behaviour of the wavefunction (or intensity matrix):

$$\psi' = \mathcal{S}^{-1} \cdot \psi \tag{2.2a}$$

$$\mathcal{I}' = \mathcal{S}^{-1} \cdot \mathcal{I} \cdot \mathcal{S} \tag{2.2b}$$

is also transferred to the densities, i.e.

$$\begin{aligned} \Delta_i'(x) &= \bar{\psi}(x) \cdot \mathcal{S} \cdot \delta_i \cdot \mathcal{S}^{-1} \cdot \psi(x) \\ &= S^j{}_i \Delta_j(x) \\ &(\mathcal{S} \cdot \delta_i \cdot \mathcal{S}^{-1} := S^j{}_i \delta_j). \end{aligned} \tag{2.3}$$

Obviously, the group $\{\mathcal{S}^j{}_i\}$ establishes some homomorphic map of the original gauge group $(\mathcal{S})$ and the subset of *'intrinsic densities'* $\{\Delta_i\}$ is the corresponding representation space. According to their very construction, the intrinsic densities may be found to be gauge dependent and in this case they cannot be equipped with a direct physical meaning. However, as is demonstrated below, the intrinsic densities may be combined with other gauge objects to form gauge invariants (*'extrinsic densities'*) and these then can be considered as physical observables (e.g. current $j_\mu(x)$ or energy–momentum density $T_{\mu\nu}(x)$).

An interesting point regarding the physical densities is the fact that the N_f-particle bundle is the Whitney sum of the corresponding single-particle bundles. This implies that the set of observables δ_i can be subdivided into two subsets, namely those which act exclusively within some single-particle subspace of the typical fibre $\mathbb{C}^{N_\mathrm{f}}$ and those which mediate between these subspaces. Let the projector onto the single-particle subspaces be denoted by $\mathcal{P}_a(x)$ $(a = 1 \ldots N_\mathrm{p})$, then the single-particle observables ${}^{(\mathrm{a})}\delta_i$ are obviously of the following form:

$$ {}^{(\mathrm{a})}\delta_i = \mathcal{P}_a \cdot \delta_i \cdot \mathcal{P}_a \tag{2.4} $$

and the corresponding single-particle densities ${}^{(\mathrm{a})}\Delta_i$ are then given in general terms by

$$ {}^{(\mathrm{a})}\Delta_i(x) = \mathrm{tr}(\mathcal{I}(x) \cdot {}^{(\mathrm{a})}\delta_i). \tag{2.5} $$

For instance, the simplest observable is, of course, the unity operator **1**

$$ \mathbf{1} = \sum_{a=1}^{N_\mathrm{f}} \mathcal{P}_a. \tag{2.6} $$

Thus we can always define the single-particle densities $\rho_a(x)$ through

$$ \rho_a(x) = \mathrm{tr}(\mathcal{I} \cdot \mathcal{P}_a). \tag{2.7} $$

Of course, the remainder of observables (not being contained in the linear vector space spanned by $\mathcal{P}_a$) must then describe the *exchange effects* occurring in our many-particle system, studied below for $N_\mathrm{f} = 2$ particles.

A further important question is related to the particle number N_p. If it is true that the proper observable objects of the theory are the extrinsic densities, how then do they display the particle number N_p of the physical system? The answer comes from the charge conservation which is usually expressed in differential form as

$$ \nabla^\mu j_\mu \equiv 0. \tag{2.8} $$

Its integral form leads to the concept of a conserved charge e

$$ e = \int_S j_\mu \, \mathrm{d}^3 S^\mu \tag{2.9} $$

where the integral extends over a three-dimensional hypersurface S (closed or open). However, for N_p particles one must demand the existence of N_p gauge-invariant conservation laws

$$ \nabla^\mu j_{a\mu} \equiv 0 \qquad (a = 1 \ldots N_\mathrm{p}) \tag{2.10} $$

with the corresponding conserved charges

$$ e_a = \int_S j_{a\mu} \, \mathrm{d}^3 S^\mu. \tag{2.11} $$

Obviously, the currents $j_{a\mu}$ present a further example for the subtle relationship between single-particle objects and the notorious exchange (or overlap) quantities. The point here is that one expects the numerical value of the charges e_a (2.11) to be independent of the particle number

N_{p}; but on the other hand the associated currents $j_{a\mu}$ (2.10) cannot be considered as strict single-particle objects because they must also contain some exchange (or overlap) constituents with respect to the other particles in order to be able, e.g., to bring forth the notorious exchange energies of atomic physics [25]. Subsequently, we will demonstrate how this problem may be solved in RST by associating the current densities $j_{a\mu}$ and the corresponding velocity operators $v_{a\mu}$ such that

$$j_{a\mu} = \mathrm{tr}(\mathcal{I} \cdot v_{a\mu}) \tag{2.12a}$$

$$(\text{or}: j_{a\mu} = \bar{\psi} \cdot v_{a\mu} \cdot \psi, \text{ resp.}). \tag{2.12b}$$

Through this construction the question of overlap effects for the densities is shifted to the exchange constituents of the operators. The problem of determining the velocity operators $v_{a\mu}$ has already been solved in a preceding paper [6] and it is sufficient here to merely reproduce the result:

$$v_{a\mu} = \frac{1}{2Mc^2}(\mathcal{P}_a \cdot \mathcal{H}_\mu + \bar{\mathcal{H}}_\mu \cdot \mathcal{P}_a). \tag{2.12c}$$

But first we have to face the problem of the right dynamics. Clearly, if the currents $j_{a\mu}$ (2.12*a*)–(2.12*c*) are required to obey the conservation laws (2.10), the intensity matrix $\mathcal{I}$ (or wavefunction ψ, resp.) must obey certain dynamical equations which admit just those conservation laws. At this point, the gauge potential $\mathcal{A}_\mu$ comes into play because the dynamics must be based upon the very successful *principle of minimal coupling* which governs all the successful modern theories of elementary particles [26]. This means that the derivatives of the wavefunction ψ (or $\mathcal{I}$, resp.) should enter the equations of motion for matter in the following gauge-covariant form:

$$\mathcal{D}_\mu \psi = \partial_\mu \psi + \mathcal{A}_\mu \cdot \psi \tag{2.13a}$$

$$\mathcal{D}_\mu \mathcal{I} = \partial_\mu \mathcal{I} + [\mathcal{A}_\mu, \mathcal{I}]. \tag{2.13b}$$

This requirement is actually realized for RST because the central equation of motion for the wavefunction ψ here is the relativistic Schrödinger equation (RSE)

$$\mathrm{i}\bar{h}c\mathcal{D}_\mu \psi = \mathcal{H}_\mu \cdot \psi \tag{2.14}$$

or, respectively, the relativistic von-Neumann equation for the intensity matrix $\mathcal{I}$

$$\mathcal{D}_\mu \mathcal{I} = \frac{\mathrm{i}}{\bar{h}c}(\mathcal{I} \cdot \bar{\mathcal{H}}_\mu - \mathcal{H}_\mu \cdot \mathcal{I}). \tag{2.15}$$

It is important that the *Hamiltonian 1-form* $\mathcal{H}_\mu$ is not Hermitian (i.e. $\bar{\mathcal{H}}_\mu \neq \mathcal{H}_\mu$). Moreover, $\mathcal{H}_\mu$ is not such an absolute object of the theory as Schrödinger's Hamiltonian operator $\hat{H}$ for the non-relativistic approach, but rather $\mathcal{H}_\mu$ obeys a certain dynamical system. The first part of this is the *conservation equation*

$$\mathcal{D}^\mu \mathcal{H}_\mu - \frac{\mathrm{i}}{\bar{h}c}\mathcal{H}^\mu \cdot \mathcal{H}_\mu = -\mathrm{i}\bar{h}c\left(\frac{Mc}{\bar{h}}\right)^2 \cdot \mathbf{1} \tag{2.16}$$

which guarantees the validity of the conservation laws (2.10) (and also the energy–momentum conservation, see [7]).

The second equation for the Hamiltonian $\mathcal{H}_\mu$ is the *integrability condition*

$$\mathcal{D}_\mu \mathcal{H}_\nu - \mathcal{D}_\nu \mathcal{H}_\mu + \frac{\mathrm{i}}{\bar{h}c}[\mathcal{H}_\mu, \mathcal{H}_\nu] = \mathrm{i}\bar{h}c\mathcal{F}_{\mu\nu} \tag{2.17}$$

where $\mathcal{F}_{\mu\nu}$ is the *field strength* due to the gauge potential $\mathcal{A}_\mu$

$$\mathcal{F}_{\mu\nu} = \nabla_\mu \mathcal{A}_\nu - \nabla_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{2.18}$$

(The field strength $\mathcal{F}_{\mu\nu}$ is the bundle curvature and therefore takes its values in the holonomy (sub)algebra of the gauge algebra

$$\mathcal{F}_{\mu\nu} = F_{a\mu\nu}\tau^a \tag{2.19}$$

where τ^a are the generators of the (sub)algebra.) The significance of the second equation for $\mathcal{H}_\mu$ (2.17) lies in the fact that it ensures the automatical validity of the bundle identities, e.g. for ψ

$$(\mathcal{D}_\mu\mathcal{D}_\nu - \mathcal{D}_\nu\mathcal{D}_\mu)\psi = \mathcal{F}_{\mu\nu}\cdot\psi \tag{2.20}$$

or similarly for $\mathcal{I}$

$$(\mathcal{D}_\mu\mathcal{D}_\nu - \mathcal{D}_\nu\mathcal{D}_\mu)\mathcal{I} = [\mathcal{F}_{\mu\nu}, \mathcal{I}]. \tag{2.21}$$

In mathematical terms, the field strength $\mathcal{F}_{\mu\nu}$ plays the part of bundle curvature and it is required to leave invariant the single-particle subspaces of the typical vector fibre [27]

$$(\mathcal{D}_\mu\mathcal{D}_\nu - \mathcal{D}_\nu\mathcal{D}_\mu)\mathcal{P}_a \equiv [\mathcal{F}_{\mu\nu}, \mathcal{P}_a] \stackrel{!}{=} 0 \tag{2.22}$$

$$(a = 1 \ldots N_{\mathrm{p}}).$$

Thus, the gauge interactions among the N_{p} particles work along the usual principles of gauge theories: the wavefunction ψ_a of the ath particle builds up the current $j_{a\mu}$ which acts as the source for the curvature [6]

$$\mathcal{D}^\mu\mathcal{F}_{\mu\nu} = 4\pi\alpha\mathcal{J}_\nu \tag{2.23}$$

$$(\mathcal{J}_\nu = j_{a\nu}\tau^a). \tag{2.24}$$

This in turn acts back on the particles via the Lorentz force density f_ν [6]

$$f_\nu = -\overline{h}c\,\mathrm{tr}(\mathcal{F}_{\mu\nu}\cdot\mathcal{J}^\mu) \tag{2.25}$$

but the single-particle subspaces are not mixed by the curvature operator $\mathcal{F}_{\mu\nu}$, cf (2.22). As a consequence, the usual gauge interactions can only produce *Coulomb interactions* among the particles but not *exchange interactions* [25]. The latter type of interactions can be explained in RST via the off-diagonal elements of the Hamiltonian $\mathcal{H}_\mu$ which mediate between the different single-particle subspaces. If $\mathcal{H}_\mu$ is diagonal in this sense, the components ψ_A of the vector section ψ (belonging to different single-particle subspaces) are disentangled and one deals with *independent particles* (corresponding to the ordinary product states in the conventional quantum theory). But if the off-diagonal elements of $\mathcal{H}_\mu$ are present in a non-trivial way, the different single-particle subspaces become coupled as shown by the RSE (2.14) ($\rightarrow$ *'entanglement'* [27]). Clearly it is one of the most interesting questions in RST, whether and in what way the exchange fields can decay to zero, i.e. the question of the transition from entanglement to independent particles. Curiously enough, it turns out that these exchange (i.e. off-diagonal) components of $\mathcal{H}_\mu$ can be factorized for a 2-particle system into strictly single-particle variables. The emergence of this possibility of factorization is one of the main concerns of our subsequent investigations because it is closely connected to the phenomenon of *renormalization*.

Finally, let us point out that within the present framework of RST the gauge interactions between matter are described in a somewhat different way in comparison with the conventional theory. In the latter approach, there is one single field strength $F_{\mu\nu}$ which is the curvature of a $U(1)$ bundle and is generated by the total electromagnetic current of the particles according to the Maxwell equations. However, in RST, the field strength $\mathcal{F}_{\mu\nu}$ is decomposed with respect to the basis $\{\tau^a\}$ of the product gauge group $U(1)\times U(1)\times\cdots\times U(1)$. Consequently the gauge potential $\mathcal{A}_\mu$ is also built up by a set of N individual gauge potentials $A_{a\mu}$ ($a = 1 \ldots N$)

$$\mathcal{A}_\mu = A_{a\mu}\tau^a \tag{2.26}$$

which are associated to the individual particles (resp. matter modes). The idea that, e.g., two particles require two gauge potentials for their description is not new but has already been put forward in connection with the problems emerging in a theory for the magnetic monopoles (see [28] and references therein). This concept of a *'multi-potential'* may appear as an ingenious solution to the monopole problems but it is an essential ingredient of the RST for *all* many-particle systems.

With these preparations, we can now go into the details of the description of general 2-particle systems where the emphasis is concentrated upon the properties of the matter subsystem (the peculiarities of the gauge-field subsystem have been studied in a preceding paper [6]).

3. Observables for two particles

As a demonstration of the concepts introduced so far, we study now the scalar 2-particle systems with electromagnetic interactions. The typical vector fibre is the two-dimensional complex space $\mathbb{C}^2$ (i.e. $N_{\mathrm{f}} = 2$) and thus the 2-component wavefunction $\psi = \{\psi_{\mathrm{a}},\ a = 1, 2\}$ is to be considered as a section of the corresponding vector bundle. There are $N_{\mathrm{f}}^2 = 4$ independent Hermitian operators acting over the typical fibre $\mathbb{C}^2$, and any other operator such as the Hamiltonian $\mathcal{H}_\mu$ or energy–momentum operator $\mathcal{T}_{\mu\nu}$ may be expanded with respect to these four basic operators. Therefore, we first have to specify the adequate operator basis.

As the first two operators we have already chosen the two projectors $\mathcal{P}_a$ $(a = 1, 2)$ which may now be complemented by two Hermitian *'permutators'* Π^a $(= g^{ab}\Pi_b)$ obeying the following relations:

$$\Pi_1 \cdot \Pi_1 = \Pi_2 \cdot \Pi_2 = \mathbf{1} \tag{3.1a}$$

$$\Pi_1 \cdot \Pi_2 = -\Pi_2 \cdot \Pi_1 = \mathrm{i}(\mathcal{P}_1 - \mathcal{P}_2) \tag{3.1b}$$

$$\Pi_1 \cdot \mathcal{P}_2 = \mathrm{i}\mathcal{P}_1 \cdot \Pi_2 = \mathcal{P}_1 \cdot \Pi_1 = \mathrm{i}\Pi_2 \cdot \mathcal{P}_2 = \tfrac{1}{2}(\Pi_1 + \mathrm{i}\Pi_2) \tag{3.1c}$$

$$\mathcal{P}_2 \cdot \Pi_1 = -\mathrm{i}\Pi_2 \cdot \mathcal{P}_1 = -\mathrm{i}\mathcal{P}_2 \cdot \Pi_2 = \Pi_1 \cdot \mathcal{P}_1 = \tfrac{1}{2}(\Pi_1 - \mathrm{i}\Pi_2) \tag{3.1d}$$

$$(\rightarrow \operatorname{tr}\Pi_a = \operatorname{tr}(\mathcal{P}_a \cdot \Pi_b) = 0).$$

From here the meaning of the permutators becomes immediately obvious through the following implications:

$$\Pi_1 \cdot \mathcal{P}_2 \cdot \Pi_1 = \Pi_2 \cdot \mathcal{P}_2 \cdot \Pi_2 = \mathcal{P}_1 \tag{3.2a}$$

$$\Pi_1 \cdot \mathcal{P}_1 \cdot \Pi_1 = \Pi_2 \cdot \mathcal{P}_1 \cdot \Pi_2 = \mathcal{P}_2 \tag{3.2b}$$

which reveal the 'permutative' properties of the operators Π_a $(a = 1, 2)$.

After the operator basis has been completed, all the observables may be decomposed with respect to this basis. For instance, the intensity matrix $\mathcal{I}$ reads

$$\mathcal{I} = \rho_a \mathcal{P}^a + \tfrac{1}{2}\, s_a \Pi^a \qquad \text{(summation of double indices)} \tag{3.3}$$

so that the densities $\{\rho_a, s_a\}$ may be recovered from $\mathcal{I}$ as

$$\rho_a = \operatorname{tr}(\mathcal{I} \cdot \mathcal{P}_a) \tag{3.4a}$$

$$s_a = \operatorname{tr}(\mathcal{I} \cdot \Pi_a). \tag{3.4b}$$

It is well known that the Fierz identity

$$\mathcal{I}^2 = \rho\mathcal{I} \qquad (\rho := \operatorname{tr}\mathcal{I} \equiv \rho_1 + \rho_2) \tag{3.5}$$

is the necessary and sufficient condition for the existence of a wavefunction ψ (i.e. $\mathcal{I} \rightarrow \psi \otimes \bar{\psi}$). This identity can now be reformulated conveniently in terms of the densities $\{\rho_a, s_a\}$ as

$$\rho_1 \cdot \rho_2 = \tfrac{1}{4} s^a s_a := \tfrac{1}{4} s^2 \qquad (s_a = s\hat{s}_a,\ \hat{s}^a\hat{s}_a = 1). \tag{3.6}$$

Observe also that the *single-particle densities* ρ_a (3.4*a*) are attributed to the previously introduced projectors $\mathcal{P}_a$ (2.6) whereas the *overlap densities* s_a (3.4*b*) are connected with the permutators Π_a. Thus one may guess that the overlap densities s_a and permutators Π_a will be responsible for the notorious exchange and overlap effects which are co-operatively produced by *both* components ψ_a ($a = 1, 2$) of the wavefunction. Indeed, in agreement with the single-particle densities (3.4*a*), putting

$$\psi_1 = a_1 \mathrm{e}^{\mathrm{i}\alpha_1} \tag{3.7a}$$

$$\psi_2 = a_2 \mathrm{e}^{\mathrm{i}\alpha_2} \tag{3.7b}$$

$$(\rightarrow \rho_1 = a_1^2,\ \rho_2 = a_2^2)$$

one finds for the overlap densities (3.4*b*) that

$$s_1 = -2a_1 a_2 \cos(\alpha_1 - \alpha_2) \tag{3.8a}$$

$$s_2 = 2a_1 a_2 \sin(\alpha_1 - \alpha_2) \tag{3.8b}$$

which concurs exactly with the Fierz identity (3.6) and, furthermore, gives a demonstration of how the overlap densities are built up by cooperation of both wavefunctions ψ_a. Evidently the overlap densities vanish wherever the wavefunctions do not overlap.

Suppose now, further, that the single-particle phases α_a change as usual according to the corresponding energy eigenvalues E_a

$$\alpha_a = -\frac{E_a t}{\hbar}. \tag{3.9}$$

Then the overlap densities s_a (3.8*a*), (3.8*b*) must rotate with the frequency ω due to the difference of the eigenvalues E_a

$$s_1 = -s \cos \omega_{\mathrm{E}} t \tag{3.10a}$$

$$s_2 = s \sin \omega_{\mathrm{E}} t \tag{3.10b}$$

$$\left(\omega_{\mathrm{E}} = \frac{E_2 - E_1}{\hbar}\right).$$

Clearly, these rapid oscillations would not be observed in a stationary system (e.g. atom) and consequently we have to consider them as a gauge artifact. But this implies that our $U(1) \times U(1)$ bundle is accompanied by some $SO(2)$ bundle which must inherit its connection from the original $U(1) \times U(1)$ connection in some definite way.

4. $SO(2)$ bundle

Consider now some $U(2)$ gauge transformation $\mathcal{S}$ which simultaneously is an element of the holonomy group $U(1) \times U(1)$:

$$\mathcal{S} = \exp(\phi_a \tau^a) \qquad (\tau^a := -\mathrm{i}\mathcal{P}^a) \tag{4.1}$$

parametrized by the group parameters ϕ_a ($a = 1, 2$). Such a change of gauge transforms any operator, e.g. the intensity matrix $\mathcal{I}$, according to the following rule:

$$\mathcal{I} \rightarrow \mathcal{I}' = \mathcal{S}^{-1} \cdot \mathcal{I} \cdot \mathcal{S}. \tag{4.2}$$

But, according to the Abelian character of this transformation, the projectors $\mathcal{P}_a$ are left invariant by $\mathcal{S}$ and so, therefore, are the single-particle densities ρ_a (3.4*a*). However, the two permutators Π_a induce some homomorphic map of the holonomy group $U(1) \times U(1)$ onto the rotation group $SO(2)$, i.e. we find by explicit computation by means of the operator algebra (3.1*a*)–(3.1*d*) that

$$\mathcal{S}^{-1} \cdot \Pi^a \cdot \mathcal{S} = S^a{}_b \Pi^b \tag{4.3}$$

with the $SO(2)$ element $\boldsymbol{S}$ being given by

$$S^a{}_b = \begin{pmatrix} \cos(\phi_1 - \phi_2) & -\sin(\phi_1 - \phi_2) \\ \sin(\phi_1 - \phi_2) & \cos(\phi_1 - \phi_2) \end{pmatrix}. \tag{4.4}$$

The overlap densities s_a are, therefore, not gauge invariant as the single-particle densities ρ_a but transform according to

$$s'_b = s_a S^a{}_b \tag{4.5}$$

with respect to a $U(1) \times U(1)$ gauge transformation. Of course the 'length' s of the 2-vector s_a, occurring in the Fierz identity (3.6), remains invariant. But the rapid oscillations (3.10*a*), (3.10*b*) can now obviously be *gauged off* by simply choosing for the gauge parameters

$$\phi_1 - \phi_2 = -\omega_{\mathrm{E}} t \tag{4.6}$$

as can always be expected for gauge artifacts.

For the sake of convenience, the $SO(2)$ gauge objects may be collected into a separate $SO(2)$ subbundle whose connection ($\boldsymbol{\omega}$) is then easily deduced from the original $\mathcal{A}_\mu$ in the following way: since the permutators Π^a have been revealed as a gauge doublet, we can put for their covariant derivative

$$\mathcal{D}_\mu \Pi_a = \Pi_b \cdot \omega^b{}_{a\mu} \qquad (\mathcal{D}_\mu \Pi_a \equiv \partial_\mu \Pi_a + [\mathcal{A}_\mu, \Pi_a]) \tag{4.7}$$

with the $\mathcal{SO}(2)$ connection $\boldsymbol{\omega}$ being given by

$$\omega^b{}_{a\mu} = -\varepsilon^b{}_a \cdot \omega_\mu. \tag{4.8}$$

Here, ε_{ab} $(= -\varepsilon_{ba})$ is the permutation tensor in two dimensions ($\mathcal{SO}(2)$ generator) and ω_μ is an ordinary 1-form. (Hint: check the consistency of the differential law (4.7) with the operator algebra (3.1*a*)–(3.1*d*) by explicit differentiation.) But now observe that the choice of such a permutator doublet Π_a is not unique. In fact, any other rotated choice Π'_a

$$\Pi'_a = \overset{\circ}{S}{}^a{}_b \cdot \Pi^b \tag{4.9}$$

with $\overset{\circ}{\boldsymbol{S}}$ an arbitrary $SO(2)$ element of the kind (4.4) would leave form invariant all the *commutators* of the set $[\mathcal{P}_a, \Pi_b]$ and $[\Pi_a, \Pi_b]$ and therefore would mediate the same bundle homomorphism $U(1) \times U(1) \to SO(2)$. With respect to the rotated doublet Π'_a (4.9) however, one must apply a modified $\mathcal{SO}(2)$ connection $\boldsymbol{\omega}'$ in (4.7):

$$\mathcal{D}_\mu \Pi'_a = \Pi'_b \cdot \omega'^b{}_{a\mu} \tag{4.10}$$

where both connections $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are related through the $SO(2)$ element $\overset{\circ}{\boldsymbol{S}}$ as usual:

$$\omega'^b{}_{a\mu} = \overset{\circ}{S}{}^b{}_c \cdot \omega^c{}_{d\mu} \cdot (\overset{\circ}{S}{}^{-1})^d{}_a + \overset{\circ}{S}{}^b{}_c \cdot \partial_\mu (\overset{\circ}{S}{}^{-1})^c{}_a. \tag{4.11}$$

Clearly the new connection $\boldsymbol{\omega}'$ is also of the form (4.8), i.e.

$$\omega'^b{}_{a\mu} = -\varepsilon^b{}_a \cdot \omega'_\mu \tag{4.12}$$

where the $\mathbb{R}^1$-valued 1-form ω_μ has been changed to

$$\omega'_\mu = \omega_\mu + \partial_\mu \zeta \tag{4.13}$$

provided the $SO(2)$ element $\overset{\circ}{\boldsymbol{S}}$ is parametrized by the group parameter ζ in the usual way

$$\overset{\circ}{S}{}^a{}_b = \cos\zeta \cdot \delta^a{}_b + \sin\zeta \cdot \varepsilon^a{}_b. \tag{4.14}$$

But the point, with such a different choice of the $SO(2)$ gauge objects, is that it does not now have any bearing upon the physical content of the formalism. Recall that the physics in fibre bundle theory is always determined by the bundle curvature $\boldsymbol{\Omega}$:

$$\Omega^a{}_{b\mu\nu} = \nabla_\mu \omega^a{}_{b\nu} - \nabla_\nu \omega^a{}_{b\mu} + \omega^a{}_{c\mu} \omega^c{}_{b\nu} - \omega^a{}_{c\nu} \omega^c{}_{b\mu}. \tag{4.15}$$

Its general form is easily computed as

$$\Omega^a{}_{b\mu\nu} = -\varepsilon^a{}_b f_{\mu\nu} \tag{4.16}$$

with the $SO(2)$ field strength $f_{\mu\nu}$ being nothing else than the curl of the 1-form ω_μ (4.8)

$$f_{\mu\nu} = \nabla_\mu \omega_\nu - \nabla_\nu \omega_\mu. \tag{4.17}$$

Consequently, the modified choice of a permutator doublet (4.9) would change only the connection coefficient ω_μ to ω'_μ (4.13) but would leave invariant the associated field strength $f_{\mu\nu}$ (4.17). It is for this reason that $f_{\mu\nu}$ can be thought to be equipped with an objective physical meaning (see below for the implications). Observe also, that a different choice of the permutator doublet $\{\Pi_a\}$ has nothing to do with a change of the $U(1) \times U(1)$ gauge, although both transformation laws (4.3) and (4.9) look quite similar. In fact, any change of the $U(1) \times U(1)$ gauge would not only transform the permutator doublet according to the homogeneous law (4.3) but would simultaneously transform the $U(1) \times U(1)$ connection $\mathcal{A}_\mu$ in the usual inhomogeneous manner

$$\mathcal{A}'_\mu = \mathcal{S}^{-1} \cdot \mathcal{A}_\mu \cdot \mathcal{S} + \mathcal{S}^{-1} \cdot \partial_\mu \mathcal{S}. \tag{4.18a}$$

As a consequence, the derivatives of the permutators are also transformed homogeneously

$$\mathcal{D}'_\mu \Pi'_a = \mathcal{S}^{-1} \cdot (\mathcal{D}_\mu \Pi_a) \cdot \mathcal{S} \tag{4.18b}$$

and if this is applied to equation (4.7) we find the $\mathcal{SO}(2)$ connection $\boldsymbol{\omega}$ (4.8) remaining invariant.

But despite this relative independency of both kinds of gauge transformations, there must exist some relationship between both curvatures $\mathcal{F}$ (2.18) and $\boldsymbol{\Omega}$ (4.16). This readily becomes clear by reconsidering the following bundle identity [7]:

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu)\Pi_a \equiv [\mathcal{F}_{\mu\nu}, \Pi_a]. \tag{4.19}$$

Introducing here the decomposition (2.19) of the curvature $\mathcal{F}_{\mu\nu}$ with respect to the holonomy generators τ^a yields on the right-hand side

$$[\mathcal{F}_{\mu\nu}, \Pi_a] = -\Pi_b \varepsilon^b{}_a f_{\mu\nu} \qquad (f_{\mu\nu} := F_{1\mu\nu} - F_{2\mu\nu}). \tag{4.20}$$

Alternatively, one can differentiate once more equation (4.7) and then one finds for the left-hand side of equation (4.19)

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu)\Pi_a = \Pi_b \Omega^b{}_{a\mu\nu} = -\Pi_b \varepsilon^b{}_a f_{\mu\nu} \tag{4.21}$$

where the curvature $\boldsymbol{\Omega}$ has already been substituted from equation (4.16). Thus, the identity (4.19) establishes the following link between both curvature 2-forms $f_{\mu\nu}$ (4.17) and $\mathcal{F}_{\mu\nu}$ (2.18):

$$f_{\mu\nu} = \tfrac{1}{2}\varepsilon_{ab} \operatorname{tr}(\Pi^b \cdot \mathcal{F}_{\mu\nu} \cdot \Pi^a). \tag{4.22}$$

From here it immediately becomes obvious that the 2-form $f_{\mu\nu}$ is actually the $SO(2)$ bundle curvature and obeys the Bianchi identity

$$\nabla_\mu f_{\nu\lambda} + \nabla_\nu f_{\lambda\mu} + \nabla_\lambda f_{\mu\nu} = 0 \tag{4.23}$$

as required.

5. Hamiltonian dynamics

Now that the geometry of the 2-particle bundle and its $SO(2)$ reduction has been determined in detail, we can turn to study the peculiar features of the *exchange fields*. These objects are connected with the permutators Π_a and therefore are responsible for the physically relevant

exchange effects. Indeed, decomposing the general (non-Hermitian!) Hamiltonian $\mathcal{H}_\mu$ with respect to the operator basis $\mathcal{P}_a$, Π_b yields

$$\mathcal{H}_\mu = H_{a\mu}\mathcal{P}^a + G_{a\mu}\Pi^a =: \hbar c(\mathcal{K}_\mu + \mathrm{i}\mathcal{L}_\mu). \tag{5.1}$$

The complex coefficients $H_{a\mu}$ ($a = 1, 2$) in front of the projectors $\mathcal{P}_a$ refer to the *single-particle* properties whereas the coefficients $G_{a\mu}$ of the permutators Π_a are expected to refer to both particles. The Hamiltonian coefficients may be further separated into their real and imaginary parts

$$H_{a\mu} = \hbar c(K_{a\mu} + \mathrm{i}L_{a\mu}) \tag{5.2a}$$

$$G_{a\mu} = \hbar c(Q_{a\mu} + \mathrm{i}N_{a\mu}) \tag{5.2b}$$

so that the kinetic field $\mathcal{K}_\mu$ (5.1) becomes

$$\mathcal{K}_\mu = K_{a\mu}\mathcal{P}^a + Q_{a\mu}\Pi^a \tag{5.3}$$

and, similarly, for the localization field

$$\mathcal{L}_\mu = L_{a\mu}\mathcal{P}^a + N_{a\mu}\Pi^a. \tag{5.4}$$

Next, we have to transcribe the abstract equations of motion for the Hamiltonian $\mathcal{H}_\mu$ to the analogous form for the coefficients just introduced. But here we have to observe that the single-particle objects (e.g. $K_{a\mu}$) are gauge invariant whereas the exchange fields (e.g. $Q_{a\mu}$) are $SO(2)$ gauge objects. Therefore, the single-particle quantities will be acted upon by the ordinary derivative (∇_μ) whereas the $SO(2)$ gauge objects must be subject to the corresponding gauge-covariant derivative ($\mathbb{D}_\mu$), e.g. for $Q_{a\mu}$

$$\mathbb{D}_\mu Q_{a\nu} := \nabla_\mu Q_{a\nu} - Q_{b\nu}\omega^b{}_{a\mu}. \tag{5.5}$$

With these arrangements the second dynamical equation for $\mathcal{H}_\mu$, namely the integrability condition (2.17), reads in coefficient form for the single-particle quantities

$$\nabla_\mu K_{1\nu} - \nabla_\nu K_{1\mu} - 2\varepsilon^{ab}(Q_{a\mu}Q_{b\nu} - N_{a\mu}N_{b\nu}) = F_{1\mu\nu} \equiv {}'F_{2\mu\nu} \tag{5.6a}$$

$$\nabla_\mu K_{2\nu} - \nabla_\nu K_{2\mu} + 2\varepsilon^{ab}(Q_{a\mu}Q_{b\nu} - N_{a\mu}N_{b\nu}) = F_{2\mu\nu} \equiv {}'F_{1\mu\nu} \tag{5.6b}$$

$$\nabla_\mu L_{1\nu} - \nabla_\nu L_{1\mu} - 2\varepsilon^{ab}(N_{a\mu}Q_{b\nu} - N_{a\nu}Q_{b\mu}) = 0 \tag{5.6c}$$

$$\nabla_\mu L_{2\nu} - \nabla_\nu L_{2\mu} + 2\varepsilon^{ab}(N_{a\mu}Q_{b\nu} - N_{a\nu}Q_{b\mu}) = 0. \tag{5.6d}$$

In a similar way, the integrability condition yields for the exchange fields

$$\mathbb{D}_\mu Q_{a\nu} - \mathbb{D}_\nu Q_{a\mu} = -\varepsilon^b{}_a(Q_{b\mu}k_\nu - Q_{b\nu}k_\mu - N_{b\mu}l_\nu + N_{b\nu}l_\mu) \tag{5.7a}$$

$$\mathbb{D}_\mu N_{a\nu} - \mathbb{D}_\nu N_{a\mu} = -\varepsilon^b{}_a(N_{b\mu}k_\nu - N_{b\nu}k_\mu + Q_{b\mu}l_\nu - Q_{b\nu}l_\mu) \tag{5.7b}$$

where the newly introduced 1-forms k_μ, l_μ are merely combinations of the corresponding single-particle objects, i.e.

$$k_\mu := \frac{\mathrm{i}}{2}\varepsilon_{ab}\,\mathrm{tr}(\Pi^b \cdot \mathcal{K}_\mu \cdot \Pi^a) = K_{1\mu} - K_{2\mu} \tag{5.8a}$$

$$l_\mu := \frac{\mathrm{i}}{2}\varepsilon_{ab}\,\mathrm{tr}(\Pi^b \cdot \mathcal{L}_\mu \cdot \Pi^a) = L_{1\mu} - L_{2\mu}. \tag{5.8b}$$

In the present context, it is not necessary to go into a detailed discusson of all the peculiar properties of these dynamical equations (5.6*a*)–(5.7*b*), but let us mention a few points. Firstly, observe that the equations (5.6*a*), (5.6*b*) for the kinetic fields ($K_{a\mu}$) define what kind of force is felt by any one of the 'particles' (i.e. modes of the matter field). Thus, the first particle (described by $K_{1\mu}$) feels the electromagnetic field ${}'F_{2\mu\nu}$ being generated by the second particle according to Maxwell's equation, but it does not feel its own field ${}'F_{1\mu\nu}$. A similar result holds for the second particle ($K_{2\mu}$). Clearly this welcome result, concerning Schrödinger's dilemma

of the missing self-interaction, is the intentional implication of our specific bundle construction for the electromagnetic interactions (for a detailed discussion of this point see [6]).

Next, observe that any one of the two particles does *not exclusively* feel the electromagnetic field generated by the other particle, but there is a third field $G_{\mu\nu}$ ($\rightarrow$ *'exchange field strength'*)

$$G_{\mu\nu} := 2\varepsilon^{ab}(Q_{a\mu}Q_{b\nu} - N_{a\mu}N_{b\nu}) \tag{5.9}$$

and this is felt by the two particles in a different way. This is immediately seen by rewriting the two curl relations (5.6*a*), (5.6*b*) in terms of $G_{\mu\nu}$ as

$$\nabla_\mu K_{1\nu} - \nabla_\nu K_{1\mu} = {}'F_{2\mu\nu} + G_{\mu\nu} \tag{5.10a}$$

$$\nabla_\mu K_{2\nu} - \nabla_\nu K_{2\mu} = {}'F_{1\mu\nu} - G_{\mu\nu}. \tag{5.10b}$$

As a consequence, it does not seem possible for both particles to occupy the same mode (i.e. $K_{1\mu} \equiv K_{2\mu}$, ${}'F_{1\mu\nu} \equiv {}'F_{2\mu\nu}$, etc), provided the exchange field strength $G_{\mu\nu}$ does not vanish. (The implications of such an 'exclusion principle' must be studied separately.) On the other hand, adding equations (5.10*a*) and (5.10*b*) gives that the total kinetic field K_μ cannot feel the exchange field strength $G_{\mu\nu}$ but only the total field $F_{\mu\nu}$

$$\nabla_\mu K_\nu - \nabla_\nu K_\mu = {}'F_{1\mu\nu} + {}'F_{2\mu\nu} \equiv F_{\mu\nu} \qquad (K_\mu := K_{1\mu} + K_{2\mu}). \tag{5.11}$$

If the 2-particle system has vanishing net charge, the total field $F_{\mu\nu}$ will decrease more rapidly than the Coulomb field and therefore the total kinetic field K_μ will just as rapidly tend to a constant 4-vector at spatial infinity ($K_\mu \rightarrow (\frac{M_* c}{\hbar}, 0, 0, 0)$, say, with M_* denoting the rest mass of the 2-particle system). Thus the rest mass M_* emerges here in the form of an eigenvalue by solving the Hamiltonian dynamics with regard of the appropriate boundary conditions at infinity (see the 1-particle example of section 9).

Finally, define the *anholonomy (tensor) field* $X_{\mu\nu}$ through

$$X_{\mu\nu} := -2\varepsilon^{ab}(Q_{a\mu}N_{b\nu} - Q_{a\nu}N_{b\mu}) \tag{5.12}$$

and then equations (5.6*c*) and (5.6*d*) are found to be of the following form:

$$\nabla_\mu L_{1\nu} - \nabla_\nu L_{1\mu} = -X_{\mu\nu} \tag{5.13a}$$

$$\nabla_\mu L_{2\nu} - \nabla_\nu L_{2\mu} = +X_{\mu\nu}. \tag{5.13b}$$

Obviously, the sum L_μ of both localization fields is a gradient field

$$\nabla_\mu L_\nu - \nabla_\nu L_\mu = 0 \qquad (L_\mu := L_{1\mu} + L_{2\mu}). \tag{5.14}$$

Therefore, one can always introduce an *'amplitude field'* $L_{\mathrm{II}}(x)$ by putting

$$L_\mu \equiv L_{1\mu} + L_{2\mu} := \frac{\partial_\mu L_{\mathrm{II}}}{L_{\mathrm{II}}}. \tag{5.15}$$

But the analogous procedure for the individual amplitudes $L_1(x)(\stackrel{?}{=}\frac{\partial_\mu L_1}{L_1})$ and $L_2(x)(\stackrel{?}{=}\frac{\partial_\mu L_2}{L_2})$ seems to be feasible only if the anholonomy tensor $X_{\mu\nu}$ (5.12) would vanish everywhere. Nevertheless, one is interested in a concept such as the individual amplitude fields because it is reasonable to assume that the singularities of the localization fields $L_{1\mu}$ and $L_{2\mu}$ are due to just the zeros of the newly introduced amplitude fields $L_1(x)$, $L_2(x)$ so that the transition to the non-singular amplitudes is actually some kind of regularization procedure.

Fortunately, it is also possible for non-vanishing $X_{\mu\nu}$ to introduce the two desired amplitude fields. To this end, observe that the general structure of the equations (5.13*a*), (5.13*b*) implies the following constraint upon the anholonomy field:

$$\nabla_\lambda X_{\mu\nu} + \nabla_\mu X_{\nu\lambda} + \nabla_\nu X_{\lambda\mu} \equiv 0 \tag{5.16}$$

which must hold over any (pseudo-)Riemannian space-time. However this Bianchi-type constraint is nothing else than the necessary and sufficient condition for the existence of an *anholonomy (vector) field* X_μ whose curl is just the corresponding tensor field $X_{\mu\nu}$:

$$X_{\mu\nu} = \nabla_\mu X_\nu - \nabla_\nu X_\mu. \tag{5.17}$$

But with this result the former integrability conditions (5.13*a*), (5.13*b*) upon the localization fields $L_{a\mu}$ can be reformulated as

$$\nabla_\mu(L_{1\nu} + X_\nu) - \nabla_\nu(L_{1\mu} + X_\mu) = 0 \tag{5.18a}$$
$$\nabla_\mu(L_{2\nu} - X_\nu) - \nabla_\nu(L_{2\mu} - X_\mu) = 0 \tag{5.18b}$$

which now admits the introduction of the desired *amplitude fields* $L_a(x)$ through

$$\frac{\partial_\mu L_1}{L_1} := L_{1\mu} + X_\mu \tag{5.19a}$$
$$\frac{\partial_\mu L_2}{L_2} := L_{2\mu} - X_\mu. \tag{5.19b}$$

Adding equations (5.19*a*) and (5.19*b*) again yields the result (5.15), but now we additionally may identify

$$L_{\mathrm{II}} = L_1 \cdot L_2. \tag{5.20}$$

The existence of two individual amplitude fields $L_a(x)$ is a pleasant result because one can show that these individual fields $L_a(x)$ must obey some second-order wave equation which, for a static situation, acquires the status of an energy-eigenvalue problem of the Schrödinger type. But in order to see this in more detail we first have to express the conservation equation in coefficient form.

Indeed, substituting the Hamiltonian decomposition (5.1)–(5.2*b*) into the conservation equation (2.16) yields the corresponding source equations for the kinetic fields

$$\nabla^\mu K_{1\mu} + 2(K_{1\mu}L_1{}^\mu + Q_{a\mu}N^{a\mu}) = 0 \tag{5.21a}$$
$$\nabla^\mu K_{2\mu} + 2(K_{2\mu}L_2{}^\mu + Q_{a\mu}N^{a\mu}) = 0 \tag{5.21b}$$

as well as for the localization fields

$$\nabla^\mu L_{1\mu} - (K_1{}^\mu K_{1\mu} - L_{1\mu}L_1{}^\mu) - (Q_{a\mu}Q^{a\mu} - N_{a\mu}N^{a\mu}) = -\left(\frac{Mc}{\hbar}\right)^2 \tag{5.22a}$$

$$\nabla^\mu L_{2\mu} - \left(K_2{}^\mu K_{2\mu} - L_{2\mu}L_2{}^\mu\right) - (Q_{a\mu}Q^{a\mu} - N_{a\mu}N^{a\mu}) = -\left(\frac{Mc}{\hbar}\right)^2. \tag{5.22b}$$

Furthermore, the analogous source equations for the exchange fields are found as

$$\mathbb{D}^\mu Q_{a\mu} + L^\mu Q_{a\mu} + K^\mu N_{a\mu} = 0 \tag{5.23a}$$
$$\mathbb{D}^\mu N_{a\mu} - (K^\mu Q_{a\mu} - L^\mu N_{a\mu}) = 0. \tag{5.23b}$$

For the moment, let us concentrate on the source equations for the localization fields $L_{a\mu}$: (5.22*a*), (5.22*b*). Observe that use of the amplitude fields $L_a(x)$ ((5.19*a*), (5.19*b*)) allows us to recast these source terms into the following form:

$$\nabla^\nu L_{1\nu} + L_{1\nu}L_1{}^\nu = \frac{\Box L_1 - 2X^\nu\partial_\nu L_1}{L_1} + X^\nu X_\nu - \nabla^\nu X_\nu \tag{5.24a}$$

$$\nabla^\nu L_{2\nu} + L_{2\nu}L_2{}^\nu = \frac{\Box L_2 + 2X^\nu\partial_\nu L_2}{L_2} + X^\nu X_\nu + \nabla^\nu X_\nu. \tag{5.24b}$$

If this is substituted now into the source equations (5.22*a*), (5.22*b*) for the localization fields, one actually arrives at the desired wave equations:

$$\Box L_1 - 2X^\nu \partial_\nu L_1 + \left[\left(\frac{Mc}{\hbar}\right)^2 - K_{1\mu}K_1{}^\mu + X^\nu X_\nu - \nabla^\nu X_\nu - Q_{a\mu}Q^{a\mu} + N_{a\mu}N^{a\mu}\right]\cdot L_1 = 0 \tag{5.25a}$$

$$\Box L_2 + 2X^\nu \partial_\nu L_2 + \left[\left(\frac{Mc}{\hbar}\right)^2 - K_{2\mu}K_2{}^\mu + X^\nu X_\nu + \nabla^\nu X_\nu - Q_{a\mu}Q^{a\mu} + N_{a\mu}N^{a\mu}\right]\cdot L_2 = 0. \tag{5.25b}$$

In order to see the analogy with the conventional eigenvalue problems of Schrödinger's non-relativistic quantum mechanics, consider a static configuration (where the four-dimensional d'Alembertian $\Box$ reduces to the Laplacian Δ) and further assume that, for spatial infinity ($r \to \infty$), all the exchange fields vanish sufficiently fast:

$$\lim_{r\to\infty}(X^\nu X_\nu \pm \nabla^\nu X_\nu + N^{a\mu}N_{a\mu} - Q^{a\mu}Q_{a\mu}) = 0. \tag{5.26}$$

Moreover, the length of the kinetic fields is expected to tend to a constant

$$\lim_{r\to\infty}(K_{a\mu}K_a{}^\mu) = \left(\frac{M_a c}{\hbar}\right)^2 \qquad M_a = \text{const.} \tag{5.27}$$

With these presumptions, the amplitude equations (5.25*a*), (5.25*b*) then adopt the following asymptotic form for spatial infinity ($r \to \infty$):

$$-\Delta L_a + \left[\left(\frac{Mc}{\hbar}\right)^2 - \left(\frac{M_a c}{\hbar}\right)^2\right]\cdot L_a = 0 \tag{5.28}$$

and will admit localized solutions with the following asymptotic behaviour:

$$L_a(x) \approx r^n \cdot \exp\left[-\sqrt{\left(\frac{Mc}{\hbar}\right)^2 - \left(\frac{M_a c}{\hbar}\right)^2}\cdot r\right]. \tag{5.29}$$

Clearly, the '*energy eigenvalues*' M_a (5.27) can quantitatively be determined only by solving the full equations (5.25*a*), (5.25*b*) and their precise numerical values will essentially depend upon the spatial characteristics of the 'potential' $-\boldsymbol{K}\cdot\boldsymbol{K} + \boldsymbol{X}\cdot\boldsymbol{X} \pm \boldsymbol{\nabla}\cdot\boldsymbol{X} - \boldsymbol{Q}\cdot\boldsymbol{Q} + \boldsymbol{N}\cdot\boldsymbol{N}$ containing the exchange fields $\boldsymbol{X}$, $\boldsymbol{N}$ and $\boldsymbol{Q}$. For the subsequent discussion of the 2-particle exchange effects one may think of those localized solutions.

Concerning the introduction of the amplitude fields $L_a(x)$ ((5.19*a*), (5.19*b*)), a final point has to be clarified about the uniqueness of the anholonomy vector X_μ. Clearly, any other vector X'_μ differing from the original X_μ by a gradient field

$$X_\mu \to X'_\mu = X_\mu + \frac{\partial_\mu \xi}{\xi} \tag{5.30}$$

would equally well generate the anholonomy tensor $X_{\mu\nu}$ (5.17). But this implies that the amplitude fields $L_a(x)$ ((5.19*a*), (5.19*b*)) must be modified to

$$L_1(x) \to L'_1(x) = \xi(x)\cdot L_1(x) \tag{5.31a}$$

$$L_2(x) \to L'_2(x) = \frac{1}{\xi(x)}\cdot L_2(x) \tag{5.31b}$$

in order that the wave equations (5.25*a*), (5.25*b*) remain form invariant (i.e. one merely has to replace $L_a(x) \to L'_a(x)$, $X_\mu \to X'_\mu$). Obviously, such a transformation of the amplitude

fields L_a and the anholonomy vector X_μ is physically irrelevant and thus there arises here some new intrinsic gauge degree of freedom (*'renormalization'*). We will readily return to the question of renormalization in conjunction with the physical densities which, of course, must be invariant with respect to the rearrangements (5.30) and (5.31*a*), (5.31*b*).

6. Density dynamics

The physical densities carry special significance in RST because they are thought to be the proper observables of the theory (albeit the corresponding theoretical framework for their measurement and observation has not yet been established.) It should also have become obvious through the preceding elaborations that in the present context the most important densities are the currents $j_{a\mu}$. Indeed, these objects are the sources for the individual gauge field modes $'F_{a\mu\nu}$. Moreover, they built up the force density f_ν, and are also responsible for the existence of the N_p charge conservation laws (here $N_\mathrm{p} = 2$).

The general form of both electromagnetic currents $j_{a\mu}$ ($a = 1, 2$) for the present 2-particle system is readily obtained by means of the recipe (2.12*a*) where the velocity operators $v_{a\mu}$ have to be substituted from equation (2.12*c*) and the intensity matrix $\mathcal{I}$ from (3.3). Thus, introducing the present 2-particle Hamiltonian $\mathcal{H}_\mu$ (5.1)–(5.2*b*) into those equations yields for the velocity operators

$$v_{1\mu} = \frac{\hbar}{Mc}\left[K_{1\mu}\cdot\mathcal{P}_1 + \frac{1}{2}(Q_{a\mu} - \varepsilon^b{}_a N_{b\mu})\cdot\Pi^a\right] \tag{6.1a}$$

$$v_{2\mu} = \frac{\hbar}{Mc}\left[K_{2\mu}\cdot\mathcal{P}_2 + \frac{1}{2}(Q_{a\mu} + \varepsilon^b{}_a N_{b\mu})\cdot\Pi^a\right]. \tag{6.1b}$$

This is an interesting result because it says that the operators $v_{a\mu}$ are additively composed of both the single-particle objects $K_{a\mu}$ and the exchange fields $Q_{a\mu}$, $N_{a\mu}$. As a consequence, the corresponding currents $j_{a\mu}$ (2.12*a*)–(2.12*c*) must receive some constituent which is due to the overlap of their wavefunctions:

$$j_{1\mu} = \frac{\hbar}{Mc}\left[\rho_1 K_{1\mu} + \frac{1}{2}s^a(Q_{a\mu} - \varepsilon^b{}_a N_{b\mu})\right] \tag{6.2a}$$

$$j_{2\mu} = \frac{\hbar}{Mc}\left[\rho_2 K_{2\mu} + \frac{1}{2}s^a(Q_{a\mu} + \varepsilon^b{}_a N_{b\mu})\right]. \tag{6.2b}$$

In order to gain some confidence in the existence of two separately conserved charges e_a (2.11) for such an entangled 2-particle system, one may check both conservation laws by explicit differentiation of the current densities $j_{a\mu}$ ((6.2*a*), (6.2*b*)). For this procedure one can use the source equations (5.21*a*)–(5.23*b*) for the kinetic fields $K_{a\mu}$, localization fields $L_{a\mu}$ and exchange fields $Q_{a\mu}$, $N_{a\mu}$. But evidently one also needs the derivation of the physical densities ρ_1, ρ_2, s_a. However, these may be directly deduced from the relativistic von Neumann equation (2.15) with the decomposition of the intensity matrix $\mathcal{I}$ being given by equation (3.3). The result is then

$$\partial_\mu\rho_1 = 2\rho_1 L_{1\mu} + s^a N_{a\mu} + \varepsilon_{ab}Q^a{}_\mu s^b \tag{6.3a}$$

$$\partial_\mu\rho_2 = 2\rho_2 L_{2\mu} + s^a N_{a\mu} - \varepsilon_{ab}Q^a{}_\mu s^b \tag{6.3b}$$

$$\mathbb{D}_\mu s_a = s_a L_\mu + \varepsilon^b{}_a s_b k_\mu + 2(\rho_1+\rho_2)N_{a\mu} - 2(\rho_1-\rho_2)\varepsilon^b{}_a Q_{b\mu}. \tag{6.3c}$$

Besides the test of the charge conservation laws, a nice consistency check consists in also differentiating the Fierz identity (3.6) and using the present density derivatives (6.3*a*)–(6.3*c*).

Here it is convenient to introduce a new variable, the *'Fierz deviation'* Δ_F [9] which is a measure for the invalidation of the Fierz identity:

$$\begin{aligned}\Delta_F &:= \mathrm{tr}(\rho \cdot \mathcal{I} - \mathcal{I}^2) \\ &= 2(\rho_1\rho_2 - \tfrac{1}{4}s^2).\end{aligned} \tag{6.4}$$

Thus the validity of the Fierz identity (3.6) is equivalent to the vanishing of the scalar deviation Δ_F. Consequently the scalar field $\Delta_F(x)$ must obey some equation of motion which admits the special solution $\Delta_F \equiv 0$ implying the pure states. Clearly, the desired equation of motion for Δ_F is found by simply forming the derivative of its definition (6.4) and substituting therein the derivatives of the physical densities from equations (6.3*a*)–(6.3*c*). This procedure yields the following result:

$$\partial_\mu \Delta_F = 2\Delta_F \cdot L_\mu \tag{6.5}$$

which is in full agreement with the gradient property (5.14) of the localization field L_μ. The solution of (6.5) expresses the Fierz deviation in terms of the amplitude fields as

$$\begin{aligned}\Delta_F(x) = \Delta_{F,\mathrm{in}} \frac{L_{II}(x)^2}{{L_{II,\mathrm{in}}}^2} = \Delta_{F,\mathrm{in}} \left(\frac{L_1(x) \cdot L_2(x)}{L_{1,\mathrm{in}} \cdot L_{2,\mathrm{in}}} \right)^2 \\ (\Delta_{F,\mathrm{in}} = \text{const.},\ L_{II,\mathrm{in}} = \text{const.},\ L_{a,\mathrm{in}} = \text{const.}).\end{aligned} \tag{6.6}$$

Thus, in a connected region of space-time, where all the amplitude fields $L_a(x)$ are regular, the Fierz deviation $\Delta_F(x)$ cannot change sign and vanishes at the zeros of $L_a(x)$. Furthermore, the pure states ($\Delta_F(x) \equiv 0$) are trivially contained in the general solution (6.6), namely for $\Delta_{F,\mathrm{in}} = 0$.

7. Factorization of the 2-particle fields

Contrary to the single-particle fields such as $K_{a\mu}, L_{a\mu}$ (5.2*a*) which are associated with a definite mode of the matter field (i.e. 'particle'), the exchange fields $Q_{a\mu}, N_{a\mu}$ (5.2*b*) are true 2-particle fields, i.e. they carry information about the inter-relationship of both particles. As a consequence, the theory so far developed relies upon the simultaneous use of 1-particle and 2-particle fields. Therefore the question arises, whether perhaps the 2-particle fields can be decomposed into some combination of single-particle fields, similarly as the overlap density s^2 (3.6) could be expressed as the product of the single-particle densities $\rho_a (a = 1, 2)$. Certainly, it is a nice task to try to reformulate the entire 2-particle theory in terms of single-particle concepts so that the 2-particle effects are hidden behind the specific coupling of the single-particle objects.

The point of departure for such a transcription is the decomposition of the exchange fields with respect to the $SO(2)$ unit vector $\hat{s}_a$ (3.6). Indeed, the existence of such a unit element gives rise to the longitudinal ($||$) and transverse ($\perp$) splitting of the exchange fields in the following way:

$$Q_{a\mu} = {}^{(||)}Q_\mu \cdot \hat{s}_a + {}^{(\perp)}Q_\mu \cdot \varepsilon^b{}_a \hat{s}_b \tag{7.1a}$$

$$N_{a\mu} = {}^{(||)}N_\mu \cdot \hat{s}_a + {}^{(\perp)}N_\mu \cdot \varepsilon^b{}_a \hat{s}_b. \tag{7.1b}$$

Clearly, this set of longitudinal and transverse components $\{{}^{(||)}Q_\mu,\ {}^{(\perp)}Q_\mu,\ {}^{(\perp)}N_\mu,\ {}^{(||)}N_\mu\}$ does not itself yet refer to the single particles, but the desired relationship becomes immediately obvious by simply introducing the present splitting ((7.1*a*), (7.1*b*)) into the density dynamics (6.3*a*)–(6.3*c*). This then yields, with the simultaneous elimination of the localization

fields $L_{a\mu}$ in favour of the amplitudes L_a ((5.19*a*), (5.19*b*)):

$$\partial_\mu\left(\frac{\rho_1}{L_1^2}\right) = -2\frac{\rho_1}{L_1^2}\,X_\mu + \frac{s}{L_1^2}({}^{(\|)}N_\mu - {}^{(\perp)}Q_\mu) \tag{7.2a}$$

$$\partial_\mu\left(\frac{\rho_2}{L_2^2}\right) = +2\frac{\rho_2}{L_2^2}\,X_\mu + \frac{s}{L_2^2}({}^{(\|)}N_\mu + {}^{(\perp)}Q_\mu) \tag{7.2b}$$

$$\partial_\mu\left(\frac{s}{L_{\mathrm{II}}}\right) = 2\frac{\rho_1}{L_{\mathrm{II}}}({}^{(\|)}N_\mu + {}^{(\perp)}Q_\mu) + 2\frac{\rho_2}{L_{\mathrm{II}}}({}^{(\|)}N_\mu - {}^{(\perp)}Q_\mu). \tag{7.2c}$$

From this form of density dynamics one can immediately express two closely related suggestions concerning the desired transcription to single-particle concepts. The first consists of the introduction of new fields $B_{a\mu}$ by putting

$$B_{1\mu} := -\frac{1}{2}\frac{L_2}{L_1}\cdot({}^{(\|)}N_\mu - {}^{(\perp)}Q_\mu) \tag{7.3a}$$

$$B_{2\mu} := -\frac{1}{2}\frac{L_1}{L_2}\cdot({}^{(\|)}N_\mu + {}^{(\perp)}Q_\mu). \tag{7.3b}$$

The second suggestion aims at the introduction of the *renormalization factors* Z_a, Z_0 which connect the amplitude fields L_a, L_{II} to the physical densities:

$$\rho_1 = Z_1\cdot L_1^2 \tag{7.4a}$$

$$\rho_2 = Z_2\cdot L_2^2 \tag{7.4b}$$

$$s = Z_0\cdot L_{\mathrm{II}} \equiv Z_0\cdot L_1\cdot L_2. \tag{7.4c}$$

With these rearrangements, the density dynamics (7.2*a*)–(7.2*c*) become

$$\partial_\mu Z_1 = -2Z_1\cdot X_\mu - 2Z_0\cdot B_{1\mu} \tag{7.5a}$$

$$\partial_\mu Z_2 = 2Z_2\cdot X_\mu - 2Z_0\cdot B_{2\mu} \tag{7.5b}$$

$$\partial_\mu Z_0 = -4Z_1 B_{2\mu} - 4Z_2\cdot B_{1\mu}. \tag{7.5c}$$

It may appear here as if the original aim, concerning a strict single-particle reformulation of the density dynamics, has not been attained completely because there still emerge the 2-particle concepts Z_0 and X_μ. However, these two objects are themselves closely related to the single-particle quantities as may be seen by the following arguments.

First, consider the question of the renormalization factor Z_0 and introduce the defining equations (7.4*a*), (7.4*b*) into the Fierz deviations (6.4) and (6.6). This yields for the Fierz identity in terms of the renormalization factors

$$Z_1\cdot Z_2 - \tfrac{1}{4}Z_0^2 = \text{const}. \tag{7.6}$$

and consequently Z_0 essentially is the product of the single-particle factors Z_a (for a pure state, the right-hand side of (7.6) is zero). Next, in order to reveal the nature of the anholonomy vector X_μ, consider the current densities $j_{a\mu}$ ((6.2*a*), (6.2*b*)) which may be expressed in terms of the longitudinal and transverse fields:

$$j_{1\mu} = \frac{\hbar}{Mc}\left[\rho_1 K_{1\mu} + \frac{1}{2}s({}^{(\|)}Q_\mu + {}^{(\perp)}N_\mu)\right] \tag{7.7a}$$

$$j_{2\mu} = \frac{\hbar}{Mc}\left[\rho_2 K_{2\mu} + \frac{1}{2}s({}^{(\|)}Q_\mu - {}^{(\perp)}N_\mu)\right]. \tag{7.7b}$$

This suggests we introduce two further single-particle fields $C_{a\mu}$ through

$$C_{1\mu} = \frac{1}{2}\frac{L_2}{L_1}\cdot({}^{(\|)}Q_\mu + {}^{(\perp)}N_\mu) \tag{7.8a}$$

$$C_{2\mu} = -\frac{1}{2}\frac{L_1}{L_2}\cdot({}^{(\|)}Q_\mu - {}^{(\perp)}N_\mu) \tag{7.8b}$$

so that the current densities $j_{a\mu}$ appear to be built up essentially from 1-particle objects:

$$j_{1\mu} = \frac{\hbar}{Mc}(Z_1 \cdot K_{1\mu} + Z_0 \cdot C_{1\mu}) \cdot L_1^2 \tag{7.9a}$$

$$j_{2\mu} = \frac{\hbar}{Mc}(Z_2 \cdot K_{2\mu} - Z_0 \cdot C_{2\mu}) \cdot L_2^2. \tag{7.9b}$$

Obviously both currents are built up by two parts where the first one is constituted exclusively by single-particle objects (i.e. $Z_a, L_a, K_{a\mu}$) and therefore would also be present for *independent* particles. However, the second part also contains the renormalization factor Z_0 which is essentially a 2-particle object, cf (7.6). Therefore, this second part must be made responsible for the overlap and exchange effects occurring in the 2-particle system (for a preliminary discussion see [25]).

Now that the exchange fields have been decomposed into single-particle objects, one would like to see also how the previous 2-particle objects of exchange field strength $G_{\mu\nu}$ (5.9) and anholonomy tensor $X_{\mu\nu}$ (5.12) look like in the single-particle formulation. The answer is very pleasant, because the amplitude fields completely drop out and the result is

$$G_{\mu\nu} = 4(B_{1\mu}C_{2\nu} - B_{1\nu}C_{2\mu} + B_{2\mu}C_{1\nu} - B_{2\nu}C_{1\mu}) \tag{7.10a}$$

$$X_{\mu\nu} = 4(C_{1\mu}C_{2\nu} - C_{1\nu}C_{2\mu} + B_{1\mu}B_{2\nu} - B_{1\nu}B_{2\mu}). \tag{7.10b}$$

Despite their apparent similarities, both tensor fields $G_{\mu\nu}$ and $X_{\mu\nu}$ are rather different objects. First, observe that for the anholonomy tensor there always exists the generating vector X_μ (5.17) which transforms *inhomogeneously* under renormalization, cf (5.30). Does such a generating vector field G_μ also exist for the exchange field strength $G_{\mu\nu}$, i.e.

$$\nabla_\mu G_\nu - \nabla_\nu G_\mu = G_{\mu\nu}. \tag{7.11}$$

Indeed, the answer is positive and the exchange vector G_μ looks as follows:

$$G_\mu = -2\frac{Z_1}{Z_0} \cdot C_{2\mu} - 2\frac{Z_2}{Z_0} \cdot C_{1\mu} \tag{7.12}$$

(for a verification of (7.12), see below). However, this vector G_μ does not stand on the same footing with the anholonomy vector X_μ. The reason is that both objects must transform differently with respect to renormalization. In order to be readily convinced of this assertion, reconsider the integrability condition for the kinetic fields $K_{a\mu}$ ((5.10*a*), (5.10*b*)). With the help of the exchange vector field (7.11), these integrability conditions can also be recast, as

$$\nabla_\mu {}'K_{1\nu} - \nabla_\nu {}'K_{1\mu} = {}'F_{2\mu\nu} \tag{7.13a}$$

$$\nabla_\mu {}'K_{2\nu} - \nabla_\nu {}'K_{2\mu} = {}'F_{1\mu\nu} \tag{7.13b}$$

where the modified kinetic fields ${}'K_{a\mu}$ emerge by absorption of the exchange vector G_μ, i.e.

$${}'K_{1\mu} = K_{1\mu} - G_\mu \tag{7.14a}$$

$${}'K_{2\mu} = K_{2\mu} + G_\mu. \tag{7.14b}$$

But since the kinetic fields $K_{a\mu}$ are invariants with respect to renormalization, the present construction ((7.14*a*), (7.14*b*)) is meaningful only if the exchange vector G_μ is also an invariant, in contrast to the anholonomy vector X_μ! On the other hand, the present implication ((7.14*a*), (7.14*b*)) of equations (5.10*a*), (5.10*b*) must be possible, because the individual field strengths ${}'F_{a\mu\nu}$ are invariants and do obey the Bianchi identities [6]

$$\nabla_\mu {}'F_{a\nu\lambda} + \nabla_\nu {}'F_{a\lambda\mu} + \nabla_\lambda {}'F_{a\mu\nu} \equiv 0. \tag{7.15}$$

Thus, the substitutions (7.14*a*), (7.14*b*) must be consistent and they express the influence of the exchange effect upon the individual kinetic fields.

Finally, one should explicitly specify the renormalization behaviour of the various objects in order to make sure that the physical quantities are actually invariant. Let us begin with the physical densities ρ_a and s ((7.4*a*)–(7.4*c*)) whose physical relevance is guaranteed by the basic philosophy of RST as a fluid-dynamic approach. Thus, the densities $\{\rho_a, s\}$ are presumed to be true invariants; and since the amplitude fields $L_a(x)$ change according to the rescaling laws (5.31*a*), (5.31*b*), it is mandatory that the renormalization factors transform as follows:

$$Z_1 \to Z_1' = \frac{Z_1}{\xi^2} \tag{7.16a}$$

$$Z_2 \to Z_2' = \xi^2 \cdot Z_2 \tag{7.16b}$$

$$Z_0 \to Z_0' \equiv Z_0. \tag{7.16c}$$

Indeed, this transformation law is just consistent with the Fierz constraint (7.6) and also ensures the invariance of the single-particle contributions to the current densities $j_{a\mu}$ ((7.9*a*), (7.9*b*)). Next, the scaling law for the new vector fields $B_{a\mu}$ and $C_{a\mu}$ is given by

$$B_{1\mu} \to B_{1\mu}' = \frac{B_{1\mu}}{\xi^2} \tag{7.17a}$$

$$B_{2\mu} \to B_{2\mu}' = \xi^2 \cdot B_{2\mu} \tag{7.17b}$$

$$C_{1\mu} \to C_{1\mu}' = \frac{1}{\xi^2} \cdot C_{1\mu} \tag{7.17c}$$

$$C_{2\mu} \to C_{2\mu}' = \xi^2 \cdot C_{2\mu}. \tag{7.17d}$$

As a consequence, both the exchange field strength $G_{\mu\nu}$ (7.10*a*) and the anholonomy tensor $X_{\mu\nu}$ (7.10*b*) are renormalization invariant and therefore they can count as true observables of the theory. The same also applies to the exchange vector G_μ (7.12) and, as expected, to the current densities $j_{a\mu}$ ((7.9*a*), (7.9*b*)). But the dynamical equations for the factors (7.5*a*)–(7.5*c*) also remain form invariant under renormalization, which is thus completely defined by equations (5.30), (5.31*a*) and (5.31*b*), (7.16*a*)–(7.16*c*) and (7.17*a*)–(7.17*d*).

Finally, it remains to look for the dynamical equations of the new fields $B_{a\mu}$ and $C_{a\mu}$, because these must be evoked if one wishes to check the curl relation (7.11) or to test the charge conservation laws (2.10) with the current densities $j_{a\mu}$ being given by their single-particle formulations ((7.9*a*), (7.9*b*)).

8. Single-particle dynamics

For a reformulation of all the equations of motion of the theory in terms of the new fields introduced in the preceding section, it is instructive to start again with the source equations for the kinetic fields, (5.21*a*), (5.21*b*). The exchange fields $Q_{a\mu}$, $N_{a\mu}$ emerging in that equation have been decomposed into their longitudinal and transverse parts, (7.1*a*), (7.1*b*), and this decomposition has been used to introduce the single-particle fields $B_{a\mu}$ ((7.3*a*), (7.3*b*)) and $C_{a\mu}$ ((7.8*a*), (7.8*b*)). Consequently, the source equations for the kinetic fields in terms of the new fields now read

$$\nabla^\mu(L_1^2 \cdot K_{1\mu}) - 2L_1^2 \cdot K_{1\mu}X^\mu = -4L_1^2\varepsilon^{ab}B_{a\mu} \cdot C_b{}^\mu \tag{8.1a}$$

$$\nabla^\mu(L_2^2 \cdot K_{2\mu}) + 2L_2^2 \cdot K_{2\mu}X^\mu = -4L_2^2\varepsilon^{ab}B_{a\mu} \cdot C_b{}^\mu. \tag{8.1b}$$

Evidently, this form of the kinetic source equations is now well-suited for the desired test of the charge conservation laws (2.10) with the current densities $j_{a\mu}$ being given by (7.9*a*), (7.9*b*).

Next, one would also like to see the wave equation for the amplitude fields $L_a(x)$ ((5.25*a*), (5.25*b*)) being expressed in terms of the new vector fields:

$$\Box L_1 - 2X^\nu \cdot \partial_\nu L_1 + \left[\left(\frac{Mc}{\hbar}\right)^2 - K_{1\mu}K_1{}^\mu - (\nabla^\nu X_\nu) + X^\nu \cdot X_\nu + 4(B_{1\mu}B_2{}^\mu + C_{1\mu}C_2{}^\mu)\right] \cdot L_1 = 0 \tag{8.2a}$$

$$\Box L_2 + 2X^\nu \cdot \partial_\nu L_2 + \left[\left(\frac{Mc}{\hbar}\right)^2 - K_{2\mu}K_2{}^\mu + (\nabla^\nu X_\nu) + X^\nu \cdot X_\nu + 4(B_{1\mu}B_2{}^\mu + C_{1\mu}C_2{}^\mu)\right] \cdot L_2 = 0. \tag{8.2b}$$

Obviously, the expected exchange effects enter these wave equations in the form of a scalar 'potential' of the kind $\pm\boldsymbol{\nabla}\cdot\boldsymbol{X} + \boldsymbol{X}\cdot\boldsymbol{X} + 4(\boldsymbol{B}_1\cdot\boldsymbol{B}_2 + \boldsymbol{C}_1\cdot\boldsymbol{C}_2)$. Of course, such an 'exchange potential' will influence the energy eigenvalues $(\frac{M_a c}{\hbar})^2 := \boldsymbol{K}_a\cdot\boldsymbol{K}_a$ as do the lengths of the kinetic fields in the case of stationary bound states. Thus, the wave equations ((8.2*a*), (8.2*b*)) present the RST analogue of the energy-eigenvalue equation in the non-relativistic Schrödinger theory.

Finally, one has to transcribe the integrability conditions (5.7*a*), (5.7*b*) and the conservation equations (5.23*a*), (5.23*b*) for the exchange fields $Q_{a\mu}$, $N_{a\mu}$ to the new fields $B_{a\mu}$, $C_{a\mu}$. The integrability conditions are then

$$\nabla_\mu B_{1\nu} - \nabla_\nu B_{1\mu} = 2(B_{1\mu}\cdot X_\nu - B_{1\nu}\cdot X_\mu) - 2(C_{1\mu}\cdot G_\nu - C_{1\nu}\cdot G_\mu) \tag{8.3a}$$

$$\nabla_\mu B_{2\nu} - \nabla_\nu B_{2\mu} = -2(B_{2\mu}\cdot X_\nu - B_{2\nu}\cdot X_\mu) - 2(C_{2\mu}\cdot G_\nu - C_{2\nu}\cdot G_\mu) \tag{8.3b}$$

$$\nabla_\mu C_{1\nu} - \nabla_\nu C_{1\mu} = 2(C_{1\mu}\cdot X_\nu - C_{1\nu}\cdot X_\mu) + 2(B_{1\mu}\cdot G_\nu - B_{1\nu}\cdot G_\mu) \tag{8.3c}$$

$$\nabla_\mu C_{2\nu} - \nabla_\nu C_{2\mu} = -2(C_{2\mu}\cdot X_\nu - C_{2\nu}\cdot X_\mu) + 2(B_{2\mu}\cdot G_\nu - B_{2\nu}\cdot G_\mu). \tag{8.3d}$$

Here it is a nice consistency check to test the form invariance of these equations with respect to renormalization by means of the rescaling laws (7.17*a*)–(7.17*d*) together with the inhomogeneous transformation for X_μ (5.30). Remember also that the exchange vector G_μ (7.12) is required to generate the exchange field strength $G_{\mu\nu}$ (7.10*a*) via the curl relation (7.11). (Use (8.3*c*), (8.3*d*) for a check.) Similar arguments also apply to the corresponding conservation equations which may be put again into the suggestive form found for the kinetic fields ((8.1*a*), (8.1*b*))

$$\nabla^\mu(L_1^2\cdot B_{1\mu}) = -2L_1^2\cdot C_{1\mu}(K_1^\mu - G^\mu) \tag{8.4a}$$

$$\nabla^\mu(L_2^2\cdot B_{2\mu}) = 2L_2^2\cdot C_{2\mu}(K_2^\mu + G^\mu) \tag{8.4b}$$

$$\nabla^\mu(L_1^2\cdot C_{1\mu}) = 2L_1^2\cdot B_{1\mu}(K_1^\mu - G^\mu) \tag{8.4c}$$

$$\nabla^\mu(L_2^2\cdot C_{2\mu}) = -2L_2^2\cdot B_{2\mu}(K_2^\mu + G^\mu). \tag{8.4d}$$

(Convince yourself that both currents $j_{a\mu}$ (7.9*a*), (7.9*b*) actually have vanishing source (2.10) by use of the latter two equations (8.4*c*), (8.4*d*).)

9. 1-particle case and the Coulomb force

In order to elucidate the relationship between the RST and the conventional quantum theory it is instructive to consider a situation which is easily manageable within both approaches: the 1-particle problem.

Here, the Hamiltonian $\mathcal{H}_\mu$ (5.1) is simplified to a $\mathbb{C}^1$-valued 1-form ($a \to 1$)

$$\mathcal{H}_\mu = \bar{h}c(K_{1\mu} + \mathrm{i}L_{1\mu}). \tag{9.1}$$

Correspondingly, the RSE (2.14) for a single particle looks very simple:

$$\partial_\mu \psi_1 + \mathrm{i}(K_{1\mu} - A_{1\mu})\psi_1 = L_{1\mu}\psi_1 \tag{9.2}$$

and is easily integrated to yield

$$\psi_1(x) = L_1(x) \cdot \mathrm{e}^{\mathrm{i}\alpha_1(x)}. \tag{9.3}$$

This result arises on account of the curl relation (5.6*a*) for a single particle:

$$\nabla_\mu K_{1\nu} - \nabla_\nu K_{1\mu} = F_{1\mu\nu} \equiv \nabla_\mu A_{1\nu} - \nabla_\nu A_{1\mu} \qquad (\Rightarrow K_{1\mu} - A_{1\mu} = -\partial_\mu \alpha_1) \tag{9.4}$$

which allows us to introduce the phase $\alpha_1(x)$. Similarly one concludes that the anholonomy tensor $X_{\mu\nu}$ of a single particle vanishes so that the curl relation for the localization field $L_{1\mu}$ (5.13*a*) allows the introduction of the amplitude field $L_1(x)$:

$$\nabla_\mu L_{1\nu} - \nabla_\nu L_{1\mu} = 0 \qquad \left(\Rightarrow L_{1\mu} = \frac{\partial_\mu L_1}{L_1}\right). \tag{9.5}$$

The wavefunction $\psi_1(x)$ (9.3), being constructed in this way, obeys the Klein–Gordon equation

$$D^\mu D_\mu \psi_1 + \left(\frac{Mc}{\bar{h}}\right)^2 \psi_1 = 0 \qquad (D_\mu \psi_1 := \partial_\mu \psi_1 - \mathrm{i}A_{1\mu}\psi_1) \tag{9.6}$$

which is immediately deduced from the RSE (2.14) in combination with the conservation equation (2.16). Thus the single-particle realization of RST is completely equivalent to the Klein–Gordon theory.

As a quick check of this assertion consider the current density j_μ ((2.12*a*)–(2.12*c*)). First, compute the velocity operator v_μ (2.12*c*) by means of the 1-particle Hamiltonian $\mathcal{H}_\mu$ (9.1) and find

$$v_{1\mu} = \frac{1}{2Mc^2}(\mathbf{1} \cdot \mathcal{H}_\mu + \bar{\mathcal{H}}_\mu \cdot \mathbf{1}) = \frac{\bar{h}}{Mc} K_{1\mu}. \tag{9.7}$$

Consequently, the prescription (2.12*a*) for constructing the current j_μ yields

$$j_{1\mu} = \frac{\bar{h}}{Mc} L_1^2 K_{1\mu}. \tag{9.8}$$

And indeed this current establishes the required charge conservation law (2.10)

$$\nabla^\mu j_{1\mu} = 0 \tag{9.9}$$

which is nothing else than the single-particle truncation of the 2-particle source equation (5.21*a*):

$$\nabla^\mu K_{1\mu} + 2K_{1\mu} \cdot L_1{}^\mu = 0. \tag{9.10}$$

To recover the conventional form for j_μ, eliminate the Hamiltonian $\mathcal{H}_\mu$ from the velocity operator $v_{1\mu}$ (9.7) by means of the RSE (2.14) and then find the well known textbook result [29]

$$j_{1\mu} = \frac{\mathrm{i}\bar{h}}{2Mc}(\overset{*}{\psi} \cdot D_\mu \psi - \psi \cdot D_\mu \overset{*}{\psi}). \tag{9.11}$$

Similar arguments also apply to the comparison of the energy–momentum density $T_{\mu\nu}$ of both approaches.

Now that the 1-particle situation is completely understood, it remains to clarify the question of how to recover this 1-particle case as a *limit* of the 2-particle situation when the second particle becomes infinitely heavy (i.e. $\frac{M_1}{M_2} \to 0$). This question has been discussed in detail for conventional quantum theory [15] but requires a separate investigation for the present RST.

Further insight into the relationship between both approaches is obtained by a comparison of the treatment of the Coulomb force problem. This may be described by a static gauge potential reading in our geometric units

$$A_{1\mu} = A_{1\mu}(r) = \frac{Z_*\alpha_*}{r}\hat{t}_\mu \qquad (\hat{t}_\mu = \{1, 0, 0, 0\}). \tag{9.12}$$

Here, Z_* is the number of charge units carried by the force centre and α_* is the fine-structure constant ($\alpha_* = \frac{e^2}{\hbar c}$). The determination of the 1-particle spectrum for this force field within the conventional approach is based upon the Klein–Gordon equation (9.6) and the result can be found in any textbook about relativistic quantum mechanics, e.g. [29]. The result for the energy eigenvalue spectrum (including the rest mass) is

$$E_{nl} = \frac{Mc^2}{\sqrt{1 + \dfrac{(Z_*\alpha_*)^2}{\left(n_* - l_* - \frac{1}{2} + \sqrt{\left(l_* + \frac{1}{2}\right)^2 - (Z_*\alpha_*)^2}\right)^2}}} \qquad (n_* = 1, 2, 3, \ldots,\ l_* = 0, 1, 2, \ldots (n_* - 1)). \tag{9.13}$$

However, within the framework of RST, the determination of the energy eigenvalues follows along a somewhat different path, especially because the non-relativistic concept of *'energy'* must be generalized to the relativistic notion of *'four-momentum'* P_μ ($:=\hbar K_\mu$). In other words, we have to determine the kinetic field $K_{1\mu}$ and the 1-particle *'energy'* E_1 is then given by

$$E_1 = M_1c^2 = c\sqrt{P_1{}^\mu P_{1\mu}} \equiv \hbar c\sqrt{K_1{}^\mu K_{1\mu}}. \tag{9.14}$$

But clearly, some caution must be taken with this definition of the *'energy'* because the curl relation for the kinetic field (9.4) says that $K_{1\mu}$ will, in general, be found to vary over space-time whereas the 1-particle energy eigenvalue E_1 is usually considered to be a *global* integration constant of the whole field configuration. Therefore, the definition for E_1 (9.14) makes sense only if it is referred to a well-defined boundary value of the kinetic field $K_{1\mu}$! In this sense we put for our present situation

$$K_{1\mu} = K_{1\mu}(r) = \frac{M_1c}{\hbar}\hat{t}_\mu + A_{1\mu}(r) \tag{9.15}$$

so that the energy defining equation (9.14) becomes meaningful as a boundary relation for spatial infinity ($r \to \infty$). This recalls the analogous situation in general relativity where the (gravitational) mass of a body (e.g. a black hole) is also defined by the asymptotic behaviour of its gravitational field at spatial infinity [30].

With such an arrangement, the 1-particle energies $E_1 = M_1c^2$ are now found in a rather straightforward manner. Inserting the kinetic field $K_{1\mu}$ (9.15) into the static form of the truncated 1-particle equation (5.22*a*) yields for the spherically symmetric solutions

$$\dot{\Lambda}_1 + \frac{2}{r}\Lambda_1 + \Lambda_1{}^2 = \left(\frac{Mc}{\hbar}\right)^2 - \left(\frac{M_1c}{\hbar} + \frac{Z_*\alpha_*}{r}\right)^2 \tag{9.16}$$

where we have put

$$\Lambda_1 := \frac{\dot{L}_1}{L_1}\ \left(\equiv L_1^{-1}\cdot\frac{\mathrm{d}L_1}{\mathrm{d}r}\right). \tag{9.17}$$

(The extension to the non-spherically symmetric solutions should be self-evident.) Now, restricting ourselves to the ground state, the obvious solution is of the form

$$\Lambda_1(r) = \lambda_0 + \frac{\lambda_1}{r} \qquad (\lambda_0, \lambda_1| = \text{const.}) \tag{9.18}$$

where the constants λ_0, λ_1 must obey the following algebraic system:

$$\lambda_0^2 = \left(\frac{Mc}{\hbar}\right)^2 - \left(\frac{M_1 c}{\hbar}\right)^2 \tag{9.19a}$$

$$\lambda_0(1+\lambda_1) = -Z_*\alpha_* \cdot \frac{M_1 c}{\hbar} \tag{9.19b}$$

$$\lambda_1(1+\lambda_1) = -(Z_*\alpha_*)^2. \tag{9.19c}$$

Such a simple system can easily be solved for the ansatz parameters λ_0, λ_1 and the 1-particle ground state energy ${}^{(0)}E_1 = {}^{(0)}M_1c^2$:

$$\lambda_0 = -\frac{M_1 c}{\hbar} \cdot \frac{2Z_*\alpha_*}{1+\sqrt{1-(2Z_*\alpha_*)^2}} \tag{9.20a}$$

$$\lambda_1 = -\tfrac{1}{2} + \sqrt{1-(2Z_*\alpha_*)^2} \tag{9.20b}$$

$${}^{(0)}E_1 = {}^{(0)}M_1c^2 = \tfrac{1}{2}Mc^2\left(1+\sqrt{1-(2Z_*\alpha_*)^2}\right). \tag{9.20c}$$

Clearly, the present result for the ground state energy $E_1^{(0)}$ (9.20*c*) is the same as predicted by the Klein–Gordon result (9.13) for $n_* = 1, l_* = 0$. The non-relativistic limit is obtained from here as usual by expanding the relativistic results with respect to the fine-structure constant α_*.

Finally, let us convince ourselves also that our RST produces the same 1-particle wavefunction ψ_1 as predicted by the conventional Klein–Gordon theory. To this end we integrate the relation (9.17) with (9.18) for the amplitude field $L_1(r)$ and find

$$L_1(r) = L_* r^{\lambda_1}\exp(\lambda_0 r) \qquad (L_* = \text{const.}). \tag{9.21}$$

Furthermore, observe that for the 1-particle situation there can arise no overlap effects and therefore the amplitude field L_1 can be identified directly with the modulus a_1 of the wavefunction ψ_1 (3.7*a*): $a_1(x) \equiv L_1(x)$. Consequently, we are left with the determination of the phase $\alpha_1(x)$. This is easily found from the fact that the difference of the kinetic field $K_{1\mu}$ and gauge potential $A_{1\mu}$ must be a pure gauge, cf (9.4). Thus, combining this with the previous ansatz (9.15) for the kinetic field immediately yields the desired phase $\alpha_1(x)$ as

$$\alpha_1 = -\int^x (K_{1\mu} - A_{1\mu})\,\mathrm{d}x^\mu = -\frac{E_1 t}{\hbar} = -\frac{M_1c^2}{\hbar}t. \tag{9.22}$$

This then casts the wavefunction $\psi_1(x)$ into the expected form

$$\psi_1(x) = L_* \mathrm{e}^{-\mathrm{i}\frac{E_1 t}{\hbar}} \cdot \frac{\exp(\lambda_0 r)}{r^{-\lambda_1}} \qquad (L_* \ldots \text{normalization constant}). \tag{9.23}$$

To obtain the non-relativistic limit of this, one neglects the rest mass M of the particle and retains only the binding energy $E_\mathrm{b} := (M - M_1)c^2$, which, of course, must yield in lowest order of the perturbation expansion (with respect to α_*) the well known Schrödinger result for the ground state:

$$E_\mathrm{b}^{(0)} \approx \frac{1}{2}\frac{MZ_*^2 e^4}{\hbar^2} \tag{9.24}$$

cf (9.20*c*). Furthermore, the non-relativistic approximations of the ansatz parameters λ_0 and λ_1 are deduced from the previous exact results (9.20*b*), (9.20*c*) as

$$\lambda_0 \approx -\frac{Mc}{\hbar}Z_*\alpha_* := -Z_*\frac{1}{a_\mathrm{B}} \tag{9.25a}$$

$$\lambda_1 \approx -(Z_*\alpha_*)^2. \tag{9.25b}$$

Therefore, the non-relativistic wavefunction ψ_1 takes the form

$$\psi_1(x) \approx L_* \mathrm{e}^{-\mathrm{i}\frac{E_b t}{\hbar}} \cdot \frac{\exp(-Z_* \frac{r}{a_B})}{r^{(Z_*\alpha_*)^2}} \qquad \left(a_B = \frac{\hbar^2}{Me^2} \ldots \text{Bohr radius}\right). \tag{9.26}$$

There is a weak singularity at the origin which, however, is *quadratic* in the coupling constant α_* and therefore would not be present in the true Schrödinger limit (which takes account of the relativistic effects only up to *first* order in α_*).

10. Discussion

Summarizing the results, it has been demonstrated that a reparametrization of the field variables for the 2-particle systems is possible so that the occurrence of exchange and overlap effects becomes elucidated. More concretely, in place of the Hamiltonian coefficient fields $\{L_{a\mu}, Q_{a\mu}, N_{a\mu}\}$, cf (5.2*a*), (5.2*b*), we have introduced the amplitude fields L_a and anholonomy vector X_μ ((5.13*a*), (5.13*b*)), the single-particle fields $B_{a\mu}$ ((7.3*a*), (7.3*b*)) and $C_{a\mu}$ ((7.8*a*), (7.8*b*)), and their descendants of anholonomy tensor $X_{\mu\nu}$ (7.10*b*), exchange field strength $G_{\mu\nu}$ (7.10*a*) and exchange vector G_μ (7.12). Furthermore, since the physical densities ρ_a, s ((3.4*a*), (3.4*b*)) have received some rival in form of the amplitude fields L_a, it has become necessary to connect both concepts through the renormalization factors Z_a, Z_0 ((7.4*a*)–(7.4*c*)). In this way, the original dynamics for the physical densities ((6.3*a*)–(6.3*c*)) had to be replaced by the wave equations for the amplitude fields L_a ((8.2*a*), (8.2*b*)) and the equations of motion for the renormalization factors ((7.5*a*)–(7.5*c*)). This change of variables brought in the effect of renormalization which, therefore, is intimately linked to the existence of a non-trivial anholonomy tensor $X_{\mu\nu}$. As a consequence, the vanishing of $X_{\mu\nu}$ must eliminate the renormalization degree of freedom, i.e. all fields become invariants in this case. Simultaneously, the anholonomy vector X_μ (5.17) can then be put to zero for the sake of simplicity so that some of the dynamical equations receive a certain simplification, e.g. the integrability condition for $B_{1\mu}$ becomes simplified to

$$\nabla_\mu B_{1\nu} - \nabla_\nu B_{1\mu} = -2(C_{1\mu} \cdot G_\nu - C_{1\nu} \cdot G_\mu) \tag{10.1}$$

or the wave equation for L_1 (8.2*a*) then becomes

$$\Box L_1 + \left[\left(\frac{Mc}{\hbar}\right)^2 - K_{1\mu}K_1{}^\mu + 4(B_{1\mu}B_2{}^\mu + C_{1\mu}C_2{}^\mu)\right] \cdot L_1 = 0. \tag{10.2}$$

A closely related example of such a simplified situation is encountered in cosmology where the well known 'cosmological principle' [30] requires all vector fields to be proportional to the Hubble flow vector (b_μ, say) of the underlying Robertson–Walker space-time. But with this proportionality (i.e. $B_{a\mu} = B_a \cdot b_\mu, C_{a\mu} = C_a \cdot b_\mu$) the anholonomy tensor $X_{\mu\nu}$ (7.10*b*) must necessarily vanish and we arrive at the present simplified situation.

It seems to us that, by means of the preceding reparametrization of the general framework, one should now be able to attack the concrete 2-particle problems, e.g. in atomic physics (in preparation).

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